Azimuthal distributions in radiative decay of polarized τ lepton

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Various distributions over the angles of the emitted photon, especially over the azimuthal angle, in the one-meson radiative decay of the polarized τ lepton, $\tau^- \to \pi^- \gamma \nu_\tau$, have been investigated. In connection with this, the photon phase space is discussed in more detail since in the case of the polarized τ lepton it is not trivial. The decay matrix element contains both the inner bremsstrahlung and the resonance (structural) contributions. The azimuthal dependence of some observables have been calculated. They are the asymmetry of the differential decay width caused by the τ lepton polarization, the Stokes parameters of the emitted photon itself and the correlation parameters describing the influence of τ-lepton polarization on the photon Stokes parameters. The numerical estimation was done in the τ lepton rest frame for arbitrary direction of the τ lepton polarization 3-vector. The vector and axial-vector form factors describing the structure-dependent part of the decay amplitude are determined using the chiral effective theory with resonances (RχT). It was found that the features of the azimuthal distributions allows to separate various terms in the spin-dependent contribution. The so-called up-down and right-left asymmetries are also calculated.

1. INTRODUCTION

In the last time, the investigation, both theoretical and experimental, of the various azimuthal asymmetries is of great interest. Experimentally these asymmetries were measured in various processes. The distribution of the azimuthal angle for the charged hadrons has been investigated in the deep inelastic positron-proton scattering at HERA [1]. The az-

imuthal asymmetry and the transverse momentum of the forward produced charged hadrons in the muon deep inelastic scattering on the deuterium target have been studied at Fermilab [2]. The azimuthal asymmetry was studied in the semi-inclusive deep inelastic scattering of 160 GeV/c muons off a transversely polarized proton or deuteron target at CERN (the COMPASS experiment) [3]. The first measurement of the Drell-Yan angular distribution, performed by NA10 Collaboration for pion-nucleon scattering, indicates a sizable azimuthal asymmetry [4, 5]. The results of the measurement of the azimuthal asymmetry in the process $e^+e^- \to q\bar{q} \to \pi\pi X$ at the BaBar, where the two pions are produced in opposite hemispheres, were presented in Ref. [6]. The results on the azimuthal asymmetry in the leptoproduction of photons on an unpolarized hydrogen target, measured at the HERMES experiment, were presented in Ref. [7]. Note that there exist the measurement not only the azimuthal asymmetries, but also the asymmetries relative to the polar angle of a particle. The forward-backward asymmetries of the Drell-Yan lepton pairs (in the dielectron and dimuon channels) were measured in the proton-proton collisions at $\sqrt{s}=7$ TeV [8] and they are consistent with the Standard Model predictions.

Theoretically, the azimuthal asymmetries in various hadron-hadron and lepton-hadron processes were investigated in a number of papers. The main goal of these studies is the elucidation of the momentum distribution of the partons in the hadrons. Since it is non-perturbative confining effect, it cannot be calculated from the first principles. Thus, they are parameterized by introducing longitudinal and transverse (the so-called intrinsic transverse momentum) momentum both in the parton distribution and fragmentation functions. These distribution functions have received much attention in the last time [9]. The non-zero intrinsic transverse momentum of partons leads to various azimuthal asymmetries in the cross section when hadron is produced in hard scattering processes. The asymmetry of pion production in the semi-inclusive deep inelastic scattering process of unpolarized charged lepton on transversely polarized nucleon target was calculated in Ref. [10]. The $cos2\phi$ azimuthal asymmetry of the unpolarized proton-antiproton Drell-Yan dilepton production process in the Z resonance region was considered in Ref. [11]. It was found that it is possible to study the spin structure of hadrons in unpolarized collision processes in Tevatron. In Ref. [6] it was suggested to measure the Collins fragmentation function in the reaction $e^+e^- \to q\bar{q} \to h_1 h_2 X$, where two hadrons are detected in opposite jets. The measurement of the nuclear dependence of the azimuthal asymmetry in unpolarized semi-inclusive deep

inelastic scattering off a various nuclei allows to obtain a valuable information about the energy loss parameter which is one of a fundamental transport parameters of hadronic matter [12]. The authors of Ref. [13] considered the forward-backward pion charge asymmetry for the $e^+e^- \to \pi^+\pi^-\gamma$ process. The asymmetry is sensitive to the mechanisms involved in the final state radiation and it provides information on the pion form factor.

In the last decade the interest to different decays of the τ lepton is stimulated by the plans for constructing SuperKEKB (Japan) and Super $c - \tau$ (Russia) facilities [14–16]. The designed luminosity (10^{35}cm$^{-2}\cdot$ s^{-1} for the Super $c - \tau$ and 10^{36}cm$^{-2}\cdot$ s^{-1} for the Super KEKB) will allow to accumulate more than 10^{10} events with τ-lepton pairs. The very high statistics of the events gives a possibility to investigate the rare decays and search for the new physics beyond Standard Model, such as the lepton flavor violation, CP violation in the leptonic sector, and so on. A review of the present status of τ physics can be found in Ref. [17]

As we see, the investigation of the various angular distributions, especially the azimuthal asymmetries, can give additional valuable information (or simplify their extraction) about the mechanisms of the reactions under the investigation. So, we apply this approach to study the angular distributions over the polar and azimuthal angles of the photon emitted in the polarized $\tau-$ lepton decay, $\tau^- \to \pi^-\gamma\nu_\tau$.

The reasons to study this decay and the short review of the papers devoted to this decay can be found in Ref. [18], where we have investigated the radiative one-meson decay of the τ lepton, $\tau^- \to \pi^-\gamma\nu_\tau$. The photon energy spectrum and the t-distribution (t is the square of the invariant mass of the pion-photon system) of the decaying unpolarized τ lepton have been calculated and the polarization effects in this decay have also been studied. The following polarization observables have been calculated in the τ lepton rest frame: the asymmetry caused by the τ lepton polarization, the Stokes parameters of the emitted photon and the spin correlation coefficients which describe the influence of the τ lepton polarization on the photon Stokes parameters. All these quantities were calculated as a functions of the photon energy or the t variable. Any distributions over the polar and azimuthal angles of the emitted photon were not considered.

In present paper we study various angular distributions in the the polarized $\tau-$ lepton decay, $\tau^- \to \pi^-\gamma\nu_\tau$. In connection with this, the photon phase space is discussed in more detail since in the case of the polarized τ lepton it is not trivial. The azimuthal dependence

of some observables have been calculated. They are the asymmetry of the differential decay width caused by the τ lepton polarization, the Stokes parameters of the emitted photon itself and the correlation parameters describing the influence of τ-lepton polarization on the photon Stokes parameters. The numerical estimation was done in the τ lepton rest frame for arbitrary direction of the τ lepton polarization 3-vector. The so-called up-down and right-left asymmetries are also calculated.

The paper is organized as follows. In Sec. 2 the matrix element of the decay $\tau^- \to \pi^- \gamma \nu_\tau$ is considered, and the definition of the basic quantities are given. Sec. 3 is devoted to the calculation of the integral right-left asymmetries as functions of the variable t. In Sec. 4.1 the photon angular phase space is analyzed in more detail. The calculation of the distributions over the photon azimuthal angle (both for the polarized and unpolarized case)is given in Sec. 4.2. The up-down differential asymmetries are calculated in Sec. 4.3. Sec. 4.4 contains the calculation of the right-left differential asymmetries. Sec. 5 contains the discussion of the obtained results and the conclusion is given in Sec. 6.

2. GENERAL FORMALISM

The main goal of our study is the investigation of various distributions over the angles of the emitted photon, especially over the azimuthal agle, in the radiative semileptonic decay of a polarized τ lepton (the emitted photon can be also polarized)

$$\tau^-(p) \to \nu_\tau(p') + \pi^-(q) + \gamma(k). \tag{1}$$

The amplitude of this decay (see Fig. 1) includes the inner bremsstrahlung contribution (IB), caused by the radiation of the τ lepton and the point-like pion (diagrams a and b), as well as the structure-dependent contribution (SD, diagram c). The SD part of the amplitude is usually described in terms of the vector and axial-vector form factors which depend on the invariant mass squared of the photon and pion, t = $(k+q)^2$. Different theoretical models have been suggested to calculate these form factors [18–24] and to derive the differential distributions over the energies and the invariant variable t in the τ lepton rest frame in the case of unpolarized and polarized τ [18, 22].

The most developed models, based on the chiral effective theory with resonances RχT, were used in Refs. [18, 21, 24]. This theory is an extension of the chiral perturbation theory
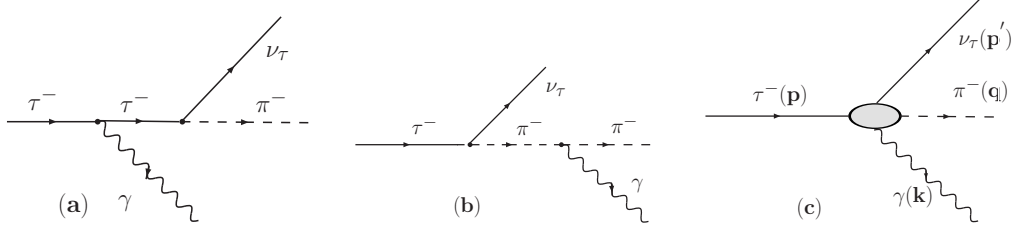
Figure 1. Feynman diagrams for the radiative $\tau^- \to \pi^- + \nu_\tau + \gamma$ decay. The diagrams a and b correspond to the so-called structure-independent inner bremsstrahlung for which it is assumed that the pion is a point-like particle. Diagram c represents the contribution of the structure-dependent part and it is parameterized in terms of the vector and axial-vector form factors.

to the region of the energies around 1 GeV, which explicitly includes the meson resonances, and has a lot applications to various aspects of the meson phenomenology [25–27].

Thus, we have for the decay amplitude

$$M_\gamma = M_{IB} + M_R,$$

$$iM_{IB} = ZM\bar{u}(p')(1+\gamma_5)\Big[\frac{\hat{k}\gamma^\mu}{2(kp)} + \frac{Ne_1^\mu}{(kp)(kq)}\Big]u(p)\varepsilon_\mu^*(k), \tag{2}$$

$$iM_R = \frac{Z}{M^2}\bar{u}(p')(1+\gamma_5)\Big\{i\gamma_\alpha(\alpha\mu kq)v(t) - \big[\gamma^\mu(qk) - q^\mu\hat{k}\big]a(t)\Big\}u(p)\varepsilon_\mu^*(k), \tag{3}$$

where $t = (k+q)^2$,

$$(\alpha\mu kq) = \epsilon^{\alpha\mu\nu\rho}k_\nu q_\rho, \quad \epsilon^{0123} = +1, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad Tr\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda = -4i\epsilon^{\mu\nu\rho\lambda}.$$

We use the same notation as in our previous work [18], namely the dimensional factor Z incorporates all constants: $Z = eG_F V_{ud} F_\pi$, M is the τ lepton mass and $\varepsilon_\mu(k)$ is the photon polarization 4-vector. Here $e^2/4\pi = \alpha = 1/137$, $G_F = 1.166 \cdot 10^{-5} GeV^{-2}$ is the Fermi constant of the weak interactions, $V_{ud} = 0.9742$ is the corresponding element of the CKM-matrix, $F_\pi = 924.42 MeV$ is the constant which determines the decay $\pi^- \to \mu^- \bar{\nu}_\mu$.

The vector $v(t)$ and axial $a(t)$ form factors in M_R amplitude read

$$a(t) = -f_A(t)\frac{M^2}{\sqrt{2}mF_\pi}, \quad v(t) = -f_V(t)\frac{M^2}{\sqrt{2}mF_\pi},$$

where $f_A(t)$ and $f_V(t)$ are

$$f_A(t) = \frac{\sqrt{2}m_{\pi\pm}}{F_\pi}\Big[\frac{F_A^2}{m_a^2 - t - im_a\Gamma_a(t)} + \frac{F_V(2G_V - F_V)}{m_\rho^2}\Big],$$

$$f_V(t) = \frac{\sqrt{2}m_{\pi\pm}}{F_\pi}\left[\frac{N_C}{24\pi^2} + \frac{4\sqrt{2}h_V F_V}{3m_\rho}\frac{t}{m_\rho^2 - t - im_\rho\Gamma_\rho(t)}\right],$$

and $\Gamma_a(t)$ ($\Gamma_\rho(t)$) is the off-mass shelf decay width of the a_1 (ρ)-meson. In our numerical calculations we use two sets of the parameters entering these form factors

	F_A	F_V	G_V
set 1	0.1368 GeV	0.1564 GeV	0.06514 GeV
set 2	F_π	$\sqrt{2}F_\pi$	$F_\pi/\sqrt{2}$

Table 1. Two sets of the coupling constants as given in [18].

We choose such normalization that the differential width of the decay (1), in terms of the matrix element M_γ, has the following form in the τ lepton rest system

$$d\Gamma = \frac{1}{4M(2\pi)^5}|M_\gamma|^2\,d\,\Phi\,,\quad d\,\Phi = \frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\delta(p'^2)\,, \tag{4}$$

where ω and ϵ are the energies of the photon and π meson. M is the τ lepton mass and the factor which corresponds to the averaging over the τ lepton spin is included in $|M_\gamma|^2$. When writing $|M_\gamma|^2$ we have to use

$$u(p)\bar{u}(p) = (\hat{p} + M)\,,\quad u(p)\bar{u}(p) = (\hat{p} + M)(1 + \gamma_5\hat{S})$$

for unpolarized and polarized τ lepton decays. Here, S is the 4-vector of τ lepton polarization.

The matrix element squared in the most general case reads

$$|M_\tau|^2 = \Sigma + \Sigma_i\,,$$

where

$$\Sigma = T^{\mu\nu}(e_{1\mu}e_{1\nu} + e_{2\mu}e_{2\nu})\,,\quad \Sigma_1 = T^{\mu\nu}(e_{1\mu}e_{2\nu} + e_{1\nu}e_{2\mu})\,,$$

$$\Sigma_2 = -i\,T^{\mu\nu}(e_{1\mu}e_{2\nu} - e_{1\nu}e_{2\mu})\,,\quad \Sigma_3 = T^{\mu\nu}(e_{1\mu}e_{1\nu} - e_{2\mu}e_{2\nu})\,.$$

Quantity Σ defines the decay width in the case of unpolarized photon, and the quantities Σ_i characterize the polarization states of the photon and can be used to define the Stokes parameters of the photon itself relative to the chosen polarization 4-vectors e_1^μ and e_2^μ. In further we use

$$e_1^\mu = \frac{1}{N}\left[(pk)q^\mu - (qk)p^\mu\right]\,,\quad e_2^\mu = \frac{(\mu pqk)}{N}\,,$$

$$N^2 = 2(qp)(pk)(qk) - M^2(qk)^2 - m^2(pk)^2 \,,$$

where m is the pion mass.

For a polarized τ lepton the current tensor is given by

$$T_{\mu\nu} = T^0_{\mu\nu} + T^S_{\mu\nu} \,,$$

where the tensor $T^S_{\mu\nu}$ depends on the τ-lepton polarization 4-vector and the tensor $T^0_{\mu\nu}$ does not dependent on it. (for the definition and analytical form of the tensor $T_{\mu\nu}$ see Ref. [18]). In this case we can write

$$\Sigma = \Sigma^0 + \Sigma^S \,, \quad \Sigma_i = \Sigma^0_i + \Sigma^S_i \,,$$

and define the physical quantities

$$A^S = \frac{\Sigma^S d\,\Phi}{\Sigma^0 d\,\Phi} \,, \quad \xi_i = \frac{\Sigma^0_i d\,\Phi}{\Sigma^0 d\,\Phi} \,, \quad \xi^S_i = \frac{\Sigma^S_i d\,\Phi}{\Sigma^0 d\,\Phi} \,, \tag{5}$$

which completely describe the polarization effects in the decay considered.

The quantity A^S is the polarization asymmetry of the differential decay width caused by the τ lepton polarization. The quantities ξ_i define the Stokes parameters of the photon itself if τ lepton is unpolarized, and the quantities ξ^S_i are the correlation parameters describing influence of the τ lepton polarization on the photon Stokes parameters.

Thus, to analyze the polarization phenomena in the process (1), we have to study both the spin-independent and spin-dependent parts of the differential width. In accordance with Eq. (4), they are

$$\frac{d\,\Gamma_0}{d\,\Phi} = g\Sigma^0 \,, \quad \frac{d\,\Gamma^S_0}{d\,\Phi} = g\Sigma^S \,, \quad \frac{d\,\Gamma_i}{d\,\Phi} = g\Sigma^0_i \,, \quad \frac{d\,\Gamma^S_i}{d\,\Phi} = g\Sigma^S_i \,, \quad g = \frac{1}{4\,M(2\,\pi)^5} \,.$$

The angular dependence in the distribution of the photon and pion in the rest system arises due to the polarization of the τ lepton through the terms (Sk), (Sq) and (Spqk)$=\epsilon_{\mu\nu\lambda\rho}S^\mu p^\nu q^\lambda k^\rho$ in the squared matrix element. The definition of the angles used is given in Fig. 2.

3. INTEGRAL RIGHT-LEFT ASYMMETRIES

The angular part of the phase space $d\,\Phi$ in Eq. (4) can be written as

$$d\,\Phi_a = \delta(c_{12} - c_1\,c_2 - s_1\,s_2\,c_\phi)d\,c_1\,d\,\phi_1\,d\,c_2\,d\,\phi_2 \,,$$
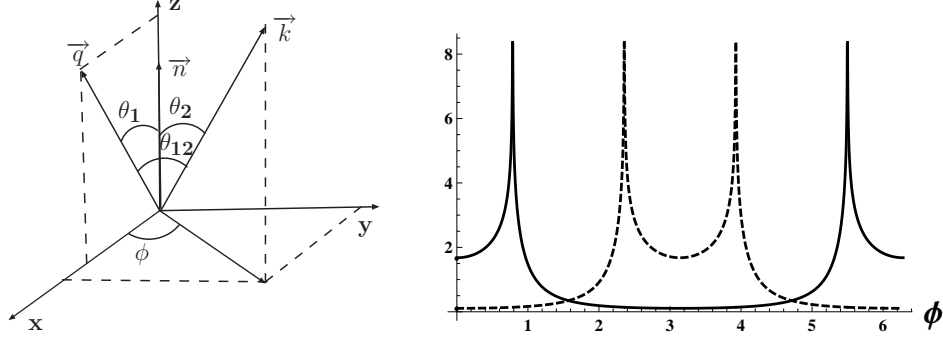
Figure 2. Definition of the angles for a polarized radiative τ decay at rest frame of the τ lepton; in this system $S = (0, \vec{n})$ (the left panel). The curves (the right panel) are the functions $I(\phi, 0.707)$ (solid line) and $I(\phi, -0.707)$ (dashed line) which are given by Eqs. (31) and (32), respectively.

where the quantity c_{12} is fixed by the energies of the photon and pion (we use notation c_i and s_i for $\cos\theta_i$ and $\sin\theta_i$).

In the case of unpolarized τ lepton, $|M_\gamma|^2$ does not depend on any angles, and we can perform the full angular integration. The most easy to do it in the system with Z axis along the direction $\mathbf{k}$ and XZ plane as $(\mathbf{k}, \mathbf{q})$ one, and the result reads

$$d\,\Phi_a = 8\,\pi^2\,.$$

Of course, this result is independent on the choice of the coordinate system. With arbitrary choice of the Z axis we can carry out one azimuthal integration and use the δ function to eliminate, for example, the second azimuthal angle. Then we receive the well known expression

$$d\,\Phi_a = 2\,\pi\,\frac{2\,dc_1\,dc_2}{K(c_1\,,c_2\,,c_{12})}\,, \quad K(c_1\,,c_2\,,c_{12}) = \sqrt{(c_1 - c_{1-})(c_{1+} - c_1)}\,, \quad c_{1\pm} = c_2\,c_{12}\pm s_2\,s_{12}\,. \quad (6)$$

The factor $2\,\pi$ in this relation reflects arbitrariness in the choosing the XZ plane, and the factor 2 in the numerator takes into account the contributions of the right $(0 < \phi < \pi)$ and left $(\pi < \phi < 2\,\pi)$ hemispheres. The function $K(c_1\,,c_2\,,c_{12})$ is symmetric relative to the change of the indexes $1 \leftrightarrows 2$.

The double angular distribution, in this coordinate system, is not trivial due to the dependence of the quantity $K(c_1\,,c_2\,,c_{12})$ on c_{12} even in the case of unpolarized τ lepton. But the single angular integration

$$\int \frac{dc_1}{K} = \int \frac{dc_2}{K} = \pi$$

eliminates this dependence and leads to full factorization of the residual angular part.

We can use such approach to describe the events corresponding to the polarized τ lepton decay choosing the coordinate system as it is shown in Fig. 2. In this case, the general form of the spin-dependent quantities Σ^S and Σ_i^S in relations (5) is very similar

$$\Sigma^S = c_1 F_1 + c_2 F_2 + s_1 s_2 s_\phi F_3\,, \quad \Sigma_i^S = c_1 G_{i1} + c_2 G_{i2} + s_1 s_2 s_\phi G_{i3}\,, \tag{7}$$

where $s_\phi = sin\phi$ and the functions F_k and G_{ik} are the angular independent ones. They depend on the pair dynamical variables (the energies of the photon and pion) which define unpolarized τ decay. The functions F_1 (G_{i1}), F_2 (G_{i2}), and F_3 (G_{i3}) are caused by the (Sq), (Sk), and (Spqk) terms, respectively. They can be obtained using the results of Ref. [18].

If, as it was done above, we use the angular δ-function to perform the full azimuthal integration, the terms, proportional to s_ϕ in (7), disappear. Further integration over the pion polar angle

$$\int \frac{dc_1}{K} = \pi\,, \quad \int \frac{c_1 dc_1}{K} = \pi c_2\, c_{12}\,, \tag{8}$$

leads to very simple angular dependence in this case

$$\int \Sigma^S \frac{dc_1}{K} = \pi\, c_2\big(c_{12}\, F_1 + F_2\big)\,, \tag{9}$$

$$\int \Sigma_i^S \frac{dc_1}{K} = \pi\, c_2\big(c_{12}\, G_{i1} + G_{i2}\big)\,.$$

Formulas (9) show that the difference of the events with the photon in the upper $(1 > c_2 > 0)$ and lower $(0 > c_2 > -1)$ hemispheres allows to single out the contribution of the spin-dependent terms (proportional to (Sq) and (Sk)) in the decay differential width. In accordance with the terminology used in our present paper, we can call them as "the integral up-down asymmetries". These effects were considered in Ref. [18].

The information, which contains in the up-down asymmetry, can be also obtained by changing the direction of the τ-lepton polarization vector $(\mathbf{n} \to -\mathbf{n})$, because at this change we have: $c_2 \to -c_2$ (since $\theta_2 \to \pi - \theta_2$). Sometimes it is preferably to detect the photons in some region of θ_2, as discussed above, than to change the direction of the τ lepton polarization vector.

Let us suppose that we performed the azimuthal integration separately in the right and left hemispheres, in such a way that

$$d\,\Phi_a = d\,\Phi_{a+}(s_\phi > 0) + d\,\Phi_{a-}(s_\phi < 0) = 2\,\pi\Big[\Big(\frac{dc_1\, dc_2}{K}\Big)_R + \Big(\frac{dc_1\, dc_2}{K}\Big)_L\Big]\,. \tag{10}$$

The difference of the events in the right and left hemispheres will be described only by the third terms in the relation (7), and the further integration of this difference with respect to c_1 and c_2 over the region

$$c_{1-} < c_1 < c_{1+} , \quad -1 < c_2 < 1$$

gives

$$\int \Sigma^S (d\,\Phi_{a+} - d\,\Phi_{a-}) = 4\,\pi^2 s_{12}\,F_3 , \quad \int \Sigma_i^S (d\,\Phi_{a+} - d\,\Phi_{a-}) = 4\,\pi^2 s_{12}\,G_{i3} . \qquad (11)$$

Thus, the corresponding measurements allow to separate the contributions caused by the term (Spqk) in the decay width. The respective effects we call as "integral right-left asymmetries".

It is clear that we can carry out the integration, in the right hand side of Eqs. (11), with respect to one of the dynamical variables and investigate the distributions over the energies $\omega\,, \epsilon$ or the invariant variable t. In the last case, the integration is performed analytically and we can write down the analytical expressions for all partial widths, which contribute to the polarization asymmetry, the Stokes parameters and the correlation parameters, in the terms of the vector and axial-vector form factors. The result reads

$$\frac{d\,\Gamma_0^{RL}}{d\,t} = \frac{P}{2}\left[Im(a(t))C_0^{RL}(t) + Im(v(t))D_0^{RL}(t)\right], \quad P = \frac{Z^2}{2^8\,\pi^3\,M^2} , \qquad (12)$$

$$C_0^{RL}(t) = \frac{8}{M(t-m^2)}\left[(t^2 + 2m^2 M^2 + m^4)J_1 - 2\,M(t+m^2)\,J_2\right],$$

$$D_0^{RL}(t) = \frac{8}{M}\left[(t+m^2)J_1 - 2\,M\,J_2\right];$$

$$\frac{d\,\Gamma_1^{RL}}{d\,t} = \frac{P}{2}\left[I_1^{RL}(t) + \left(|a(t)|^2 - |v(t)|^2\right)A_1^{RL}(t) + Re(a(t))C_1^{RL}(t) + Re(v(t))D_1^{RL}(t)\right], \quad (13)$$

$$I_1^{RL}(t) = \frac{16\,M^3}{t-m^2}\left[-J_1 + (t-m^2)J_3\right], \quad A_1^{RL}(t) = -\frac{4(t-m^2)}{M^3}\left[(t+M^2)J_1 - 2\,M\,J_2\right],$$

$$C_1^{RL}(t) = -16\left(M\,J_1 - J_2\right), \quad D_1^{RL}(t) = \frac{16}{M(t-m^2)}\left[m^2(M^2+t)J_1 - M(t+m^2)J_2\right];$$

$$\frac{d\,\Gamma_2^{RL}}{d\,t} = \frac{P}{2}\left[Im(a(t))C_2^{RL}(t) + Im(v(t))D_2^{RL}(t)\right], \qquad (14)$$

$$C_2^{RL}(t) = -D_0^{RL}(t), \quad D_2^{RL}(t) = -C_0^{RL}(t);$$

$$\frac{d\,\Gamma_3^{RL}}{d\,t} = \frac{P}{2}\left[Im(a^*(t)v(t))B_3^{RL}(t) + Im(a(t))C_3^{RL}(t) + Im(v(t))D_3^{RL}(t)\right], \qquad (15)$$

$$B_3^{RL}(t) = 2\,A_1^{RL}(t)\,, \quad C_3^{RL}(t) = D_1^{RL}(t)\,, \quad D_3^{RL}(t) = C_1^{RL}(t)\,.$$

The quantities J_i, $i = 1\,,2\,,3$, depend on the variable t and they are defined as follows

$$J_1 = \int\limits_{\omega_{min}}^{\omega_{max}} |\mathbf{q}|\, s_{12}\, d\omega\,, \quad J_2 = \int\limits_{\omega_{min}}^{\omega_{max}} \left(\frac{M^2 + t}{2\,M} - \omega\right) |\mathbf{q}|\, s_{12}\, d\omega\,, \quad J_3 = \frac{1}{2\,M} \int\limits_{\omega_{min}}^{\omega_{max}} \frac{|\mathbf{q}|}{\omega}\, s_{12}\, d\omega\,, \quad (16)$$

where $\omega_{min} = (t - m^2)/2M$ and $\omega_{max} = M(t - m^2)/2t$. The interval of the variable t is the following: $m^2 \le t \le M^2$.

The analytical form of these integrals is very simple, namely

$$J_1 = \frac{\pi\,(M^2 - t)^2(t - m^2)}{4\,M\sqrt{t}(M + \sqrt{t})^2}\,, J_2 = \frac{M^2 + t}{2\,M}\, J_1 - \frac{\pi\,(M^2 - t)^2(t - m^2)^2}{32\,M^2\,t\,\sqrt{t}}\,, J_3 = \frac{\pi\,(M^2 - t)^2}{4\,M^2(M + \sqrt{t})^2}\,.$$

In Fig. 3 we show the t-dependence of some quantities, which illustrate the integrated, over the azimuthal angle, right-left asymmetries. Together with the decay width, defined by Eq. (12), we present the right-left asymmetry $A^{RL}(t)$ and the correlation parameters $\xi_i^{RL}(t)$ defined as

$$A^{RL}(t) = \frac{d\,\Gamma_0^{RL}}{d\,t}\Big/\frac{d\,\Gamma_0}{d\,t}\,, \quad \xi_i^{RL}(t) = \frac{d\,\Gamma_i^{RL}}{d\,t}\Big/\frac{d\,\Gamma_0}{d\,t}\,, \quad (17)$$

where the expression for the unpolarized differential decay width $d\Gamma_0/dt$ is defined by Eq. (55) in Ref. [18]. Remind that the right-left asymmetries vanish for unpolarized τ lepton.

4.   DIFFERENTIAL AZIMUTHAL UP-DOWN AND RIGHT-LEFT ASYMMETRIES

4.1.   Angular phase space of the photon

The main goal of this paper is to analyze the differential distributions over the azimuthal angle ϕ including the up-down and right-left asymmetries caused by the τ lepton polarization. In this case we have to use the δ - function in the angular phase space $d\,\Phi_a$ to do the integration with respect to θ_1 (or θ_2.) This procedure leads to a more complicated angular part of the phase space

$$\frac{d\,\Phi_a}{2\pi} = dc_2\, d\phi\left[\frac{\delta(c_1 - c_+)dc_1}{|c_2 - c_+\, s_2 \cos\phi/s_+|} + \frac{\delta(c_1 - c_-)dc_1}{|c_2 - c_-\, s_2 \cos\phi/s_-|}\right]\,, \quad (18)$$
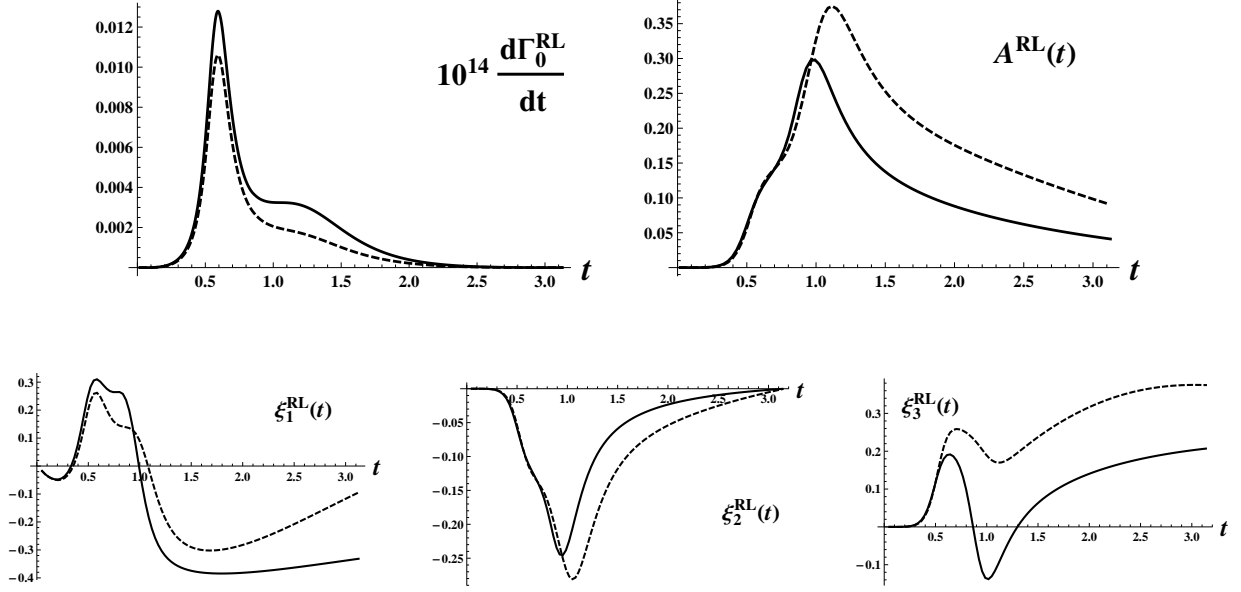
Figure 3. The difference of the differential widths, as it is defined by Eq (12), in the right and left semispheres relative to the plane $(\mathbf{n}, \mathbf{q})$, in GeV^{-1} and the variable t is given in GeV^2. The right-left integrated asymmetry and the correlation parameters defined by Eq. (17). The solid curves correspond to the set 1 of the parameters, used for description of the vector and axial-vector form factors in Ref. [18], and the dashed one – to the set 2.

where $c_\pm$ are the solutions of the equation $c_{12} = c_1 c_2 + s_1 s_2 \cos\phi$ at fixed values of c_{12} which are determined by any pair of the variables $(\epsilon, \omega), (\epsilon, t)$ or (ω, t)

$$c_\pm = \frac{1}{c_2^2 + s_2^2 \cos^2\phi}\left(c_2 c_{12} \pm s_2 \cos\phi\, Y\right), \quad Y = \sqrt{(c_2^2 + s_2^2 \cos^2\phi - c_{12}^2)}.$$

For the further calculations we need also the quantities

$$s_\pm = \frac{1}{c_2^2 + s_2^2 \cos^2\phi}\left|c_2\, Y \mp s_2\, c_{12}\, \cos\phi\right|.$$

The angular integration region, in this case, is more complex and it is specified by the conditions

$$c_2^2 + s_2^2 \cos^2\phi - c_{12}^2 > 0\,; \; (c_{12} - c_\pm c_2 > 0\,, \; \cos\phi > 0\,); \; (c_{12} - c_\pm c_2 < 0\,, \; \cos\phi < 0\,). \quad (19)$$

The entire region of the integration is divided into four parts depending on the choice between $c_1 = c_+$ and $c_1 = c_-$ and the values of $c_{12} > 0$ or $c_{12} < 0$.

The boundaries in the case $c_{12} > 0$ can be written as

$$\left[0 < \phi < \theta_{12}\,, \; 2\pi - \theta_{12} < \phi < 2\pi\,, \; -1 < c_2 < c_{12}\,, \; if \; c_1 = c_+\,, \; -c_{12} < c_2 < 1\,, \; if \; c_1 = c_-\right];$$

$$\big[\theta_{12} < \phi < \pi/2\,,\ 3\pi/2 < \phi\ < 2\pi - \theta_{12}\,,\ -1 < c_2 < -X\,,\ X < c_2 < c_{12}\,,\ if\ c_1 = c_+\,,$$

$$-c_{12} < c_2 < -X\,,\ X < c_2 < 1\,,\ if\ c_1 = c_-\,\big]\,;\ X = \sqrt{1 - \frac{s_{12}^2}{\sin^2\phi}}\,,$$

$$\big[\pi/2 < \phi < 3\pi/2\,,\ \ -1 < c_2 < -c_{12}\,,\ if\ \ c_1 = c_+\,,\ \ c_{12} < c_2 < 1\,,\ if\ c_1 = c_-\,\big]. \tag{20}$$

For $c_{12} < 0$ we have

$$\big[0 < \phi < \pi/2\,,\ 3\pi/2 < \phi < 2\pi\,,\ \ -1 < c_2 < c_{12}\,,\ if\ c_1 = c_+\,,\ \ c_{12} < c_2 < 1\,,\ if\ \ c_1 = c_-\,\big]\,;$$

$$\big[\pi/2 < \phi < \theta_{12}\,,\ \ 2\pi - \theta_{12} < \phi < 3\pi/2\,,\ \ -1 < c_2 < -X\,,\ X < c_2- < c_{12}\,,\ if\ c_1 = c_+\,,$$

$$c_{12} < c_2 < -X\,,\ \ X < c_2 < 1\ if\ c_1 = c_-\,\big]\,;$$

$$\big[\theta_{12} < \phi < 2\pi - \theta_{12}\,,\ \ -1 < c_2 < c_{12}\,,\ if\ c_1 = c_+\,,\ \ -c_{12} < c_2 < 1\,,\ if\ c_1 = c_-\,\big]. \tag{21}$$

The corresponding plots for the angular phase space in terms of the angles ϕ and θ_2 are shown in Fig. 4.

We can verify that for the ranges of the angular variables, defined by the inequalities (20) and (21) for both cases $c_{12} > 0$ and $c_{12} < 0$, the following relations always take place

$$|c_2\,Y - s_2\,c_{12}\,\cos\phi| = s_2\,c_{12}\,\cos\phi - c_2\,Y\,,$$

if we choose $c_1 = c_+$ and $s_1 = s_+$, and

$$|c_2\,Y + s_2\,c_{12}\,\cos\phi| = s_2\,c_{12}\,\cos\phi + c_2\,Y\,,$$

for $c_1 = c_-$ and $s_1 = s_-$. Therefore, we can rewrite the angular phase space in the following form

$$\delta(c_{12} - c_1\,c_2 - s_1\,s_2\,\cos\phi)d\,c_2\,d\,c_1\,d\,\phi = \bar{\bar{\Phi}}_a\,d\,c_2\,d\,\phi\,, \tag{22}$$

$$\bar{\bar{\Phi}}_a = d\,c_1\Big[\frac{\delta(c_1 - c_+)(s_2\,c_{12}\,\cos\phi - c_2\,Y)}{Y\,(c_2^2 + s_2^2\,\cos^2\phi)} + \frac{\delta(c_1 - c_-)(s_2\,c_{12}\,\cos\phi + c_2\,Y)}{Y\,(c_2^2 + s_2^2\,\cos^2\phi)}\Big]\,.$$

To be sure, we have to check that the integration over the entire angular phase space, at arbitrary values of the c_{12}, results in $4\,\pi$. Firstly note that, if $c_{12} = 0$, such integration reduces to

$$\int \bar{\bar{\Phi}}_a(c_{12} = 0)dc_2 d\phi = 2\int\limits_0^{2\pi} d\,\phi \int\limits_0^1 \frac{c_2\,d\,c_2}{c_2^2 + s_2^2\,\cos^2\phi} = -4 \int\limits_0^{\pi/2} \frac{\ln(\cos^2\phi)}{\sin^2\phi}\,d\,\phi = 4\,\pi\,.$$
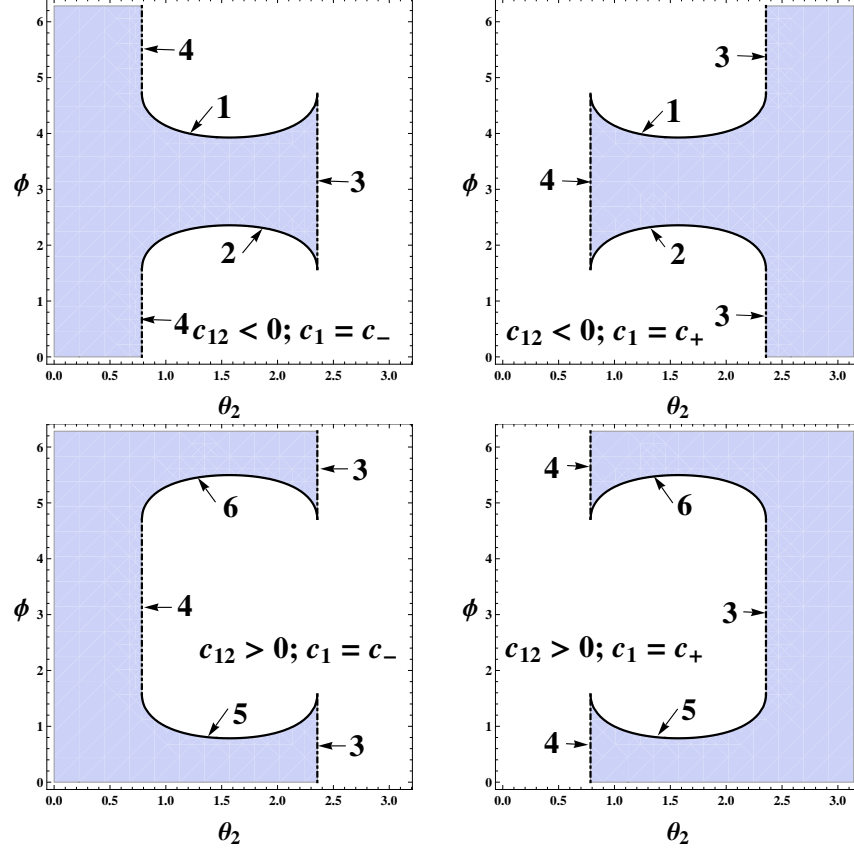
Figure 4. Four parts of the angular phase space are given in terms of the azimuthal ϕ and polar θ_2 angles of the photon. Only the shaded regions are permitted. On the lines **4** and **3** $c_2 = \pm|c_{12}|$, respectively. The lines **1** and **2** corresponds to $\phi = \pi \pm y$; on the line **5** $\phi = y$ and on the line **6** $\phi = 2\pi - y$. The quantity y is defined in Eq. (23).

Let us investigate further, for example, the case $c_{12} < 0$. After simple algebraic manipulations we can write

$$\int \bar{\bar{\Phi}}_a dc_2 d\phi = 2 \int\limits_{-c_{12}}^{1} \left\{ \int\limits_{0}^{2\pi} \frac{c_2\, d\phi}{c_2^2 + s_2^2 \cos^2 \phi} \right\} d c_2 + 2 \int\limits_{c_{12}}^{-c_{12}} \left\{ \int\limits_{\pi-y}^{\pi+y} \frac{s_2\, c_{12}\, \cos \phi\, d\phi}{Y(c_2^2 + s_2^2 \cos^2 \phi)} \right\} d c_2 , \quad (23)$$

$$y = Arcsin\left(\frac{s_{12}}{s_2}\right) .$$

The integration with respect to the azimuthal angle inside the braces in (23) gives a value 2π for the first contribution in the right hand side and π for the second one. Then we obtain

$$\Phi_a = 4\pi(1 + c_{12}) - 4\pi c_{12} = 4\pi .$$

The same result is valid, of course, in the case $c_{12} > 0$.

4.2. Integration over c_2

To investigate the single azimuthal distributions, we have to perform the integration with respect to c_2. Because the decay matrix element squared contains the contribution that does not depend on any angles, and the contributions which are proportional to c_1 (due to the term (Sq)), to c_2 (due to the term (Sk)), and to $s_1 s_2 \sin \phi$ (due to the term $(Spqk)$), the following integrals have to be evaluated

$$\int \bar{\Phi}_a \, d \, c_2 \left(1 \, , c_1 \, , c_2 \, , s_1 \, s_2 \, s_\phi \right) .$$

The values of the corresponding integrals, with c_1 and c_2 as integrands, are opposite in sign in the upper $(c_2 > 0)$ and lower $(c_2 < 0)$ hemispheres, whereas the integral with the integrand $(s_1 s_2 \sin \phi)$ is opposite in sign in the right $(\phi < \pi)$ and left $(\phi > \pi)$ hemispheres. Thus, we can extract the contribution due to the terms proportional to (Sq) and (Sk) in the matrix element squared by taking the difference of the events number in the upper and lower hemispheres and the term proportional to $(Spqk)$ – in the right and left ones. The events number for unpolarized τ lepton is the same inside all the hemispheres. In further we will normalize the different asymmetries and the correlation parameters by the corresponding unpolarized event numbers.

In spite of the nontrivial form of the phase space factor, the integration over the c_2 variable can be performed analytically. The necessary integrals are

$$I_{c_1}(\phi, \, c_{12}) = \int\limits_0^1 c_1 \, d \, c_2 \bar{\Phi}_a \, , \quad I_{c_2}(\phi, \, c_{12}) = \int\limits_0^1 c_2 \, d \, c_2 \bar{\Phi}_a \, ,$$

$$I(\phi, \, c_{12}) = \int\limits_{-1}^1 d \, c_2 \, \bar{\Phi}_a \, , \quad I_\phi(\phi, \, c_{12}) = \int\limits_{-1}^1 s_1 \, s_2 \, s_\phi \, d \, c_2 \bar{\Phi}_a \, .$$

When integrating, we have to take into account the ranges of the variables c_2 and ϕ given in Fig. 4 and consider the cases $c_{12} > 0$ and $c_{12} < 0$ separately. Thus, we have

$$s_\phi^3 \, I_{c_1}(\phi, \, c_{12} > 0) = (s_\phi - \phi \, c_\phi)(1 - c_{12}) + 2 \, c_\phi \, W_1 - 2 \sqrt{c_\phi^2 - c_{12}^2} \, \tan \phi \, , \; 0 < \phi < \theta_{12} \, , \quad (24)$$

$$[(\pi - \phi) c_\phi + s_\phi](1 - c_{12}) \, , \quad \theta_{12} < \phi < 2 \, \pi - \theta_{12} \, ,$$

$$(s_\phi + (2 \, \pi - \phi) \, c_\phi)(1 - c_{12}) + 2 \, c_\phi \, W_1 - 2 \tan \phi \, \sqrt{c_\phi^2 - c_{12}^2} \, , \quad 2 \, \pi - \theta_{12} < \phi < 2 \, \pi \, ,$$

$$W_1 = \arctan x - c_{12} \arctan (c_{12} \, x) \,, \quad x = \frac{s_\phi}{\sqrt{c_\phi^2 - c_{12}^2}} \,.$$

For the case $c_{12} < 0$ we have

$$s_\phi^3 \, I_{c_1}(\phi, \, c_{12} < 0) = (1 + c_{12})(\phi \, c_\phi - s_\phi) \,, \quad 0 < \phi < \theta_{12} \,, \tag{25}$$

$$-[s_\phi + (\pi - \phi) \, c_\phi](1 + c_{12}) + 2 \, c_\phi \, W_1 - 2 \tan \phi \, \sqrt{c_\phi^2 - c_{12}^2} \,, \quad \theta_{12} < \phi < 2\,\pi - \theta_{12} \,,$$

$$-[s_\phi + (2\,\pi - \phi) \, c_\phi](1 + c_{12}) \,, \quad 2\,\pi - \theta_{12} < \phi < 2\,\pi \,.$$

It is obvious that in the case $c_{12} = 0$ the functions $I_{c_1}(\phi, \, c_{12} > 0)$ and $I_{c_1}(\phi, \, c_{12} < 0)$ have to coincide. This can be seen using the relations

$$\arctan (\tan x) = \left(\begin{array}{ll} x, & 0 < x < \frac{\pi}{2}; \\ x - \pi, & \frac{\pi}{2} < x < \frac{3\pi}{2}; \\ x - 2\,\pi, & \frac{3\pi}{2} < x < 2\,\pi \end{array} \right) \,.$$

Let us write down analogous formulas for the quantity $I_{c_2}(\phi, c_{12})$. In the case of $c_{12} > 0$

$$s_\phi^3 \, I_{c_2}(\phi, c_{12} > 0) = \left(\begin{array}{ll} (s_\phi - \phi \, c_\phi)(1 - c_{12}) + 2 c_\phi \, W_2 \,, & 0 < \phi < \theta_{12} \\ [(\pi - \phi)c_\phi + s_\phi](1 - c_{12}) \,, & \theta_{12} < \phi < 2\,\pi - \theta_{12} \\ [2\,\pi - \phi)c_\phi + s_\phi](1 - c_{12}) + 2 c_\phi \, W_2 \,, & 2\,\pi - \theta_{12} < \phi < 2\,\pi \end{array} \right) \,, \tag{26}$$

$$W_2 = \arctan(c_{12} \, x) - c_{12} \arctan(x) \,.$$

At the negative values of the c_{12} we can write

$$s_\phi^3 \, I_{c_2}(\phi, c_{12} < 0) = \left(\begin{array}{ll} (s_\phi - \phi \, c_\phi)(1 + c_{12}) \,, & 0 < \phi < \theta_{12} \\ [(\pi - \phi)c_\phi + s_\phi](1 + c_{12}) + 2 c_\phi \, W_2 \,, & \theta_{12} < \phi < 2\,\pi - \theta_{12} \\ [2\,\pi - \phi)c_\phi + s_\phi](1 + c_{12}) \,, & 2\,\pi - \theta_{12} < \phi < 2\,\pi \end{array} \right) \,. \tag{27}$$

Again, we see that at $c_{12} = 0$ the expressions (26) and (27) coincide because in this case $W_2 = 0$.

To investigate the differential right-left effects, it is enough to calculate the quantity $I_\phi(\phi, c_{12})$ when the azimuthal angle $0 < \phi < \pi$. We can write down it in terms of the standard elliptic functions

$$I_\phi(\phi, c_{12} > 0) = \frac{2 \, c_{12} \, c_\phi}{s_\phi^3} \ln(c_\phi^2 + c_{12}^2 s_\phi^2) +$$

$$
\begin{pmatrix}
F1(\phi)\,, & 0 < \phi < \theta_{12} \\
4\,s_{12}\,\tan\theta_{12}\,\cot\phi\,\csc^2\phi - F2(\phi)\,, & \theta_{12} < \phi < \pi - \theta_{12} \\
F1(\phi)\,, & \pi - \theta_{12} < \phi < \pi
\end{pmatrix}.
\tag{28}
$$

The function $F1(\phi)$ is defined as follows

$$
F1(\phi) = \frac{2\,s_{12}}{s_\phi}\left\{ K(z) + F\big(v \mid z\big) - \frac{2}{s_\phi^2}\big[E(z) + E\big(v \mid z\big)\big] \right\} -
$$

$$
\frac{4(c_{12}^2 - c_\phi^2)}{s_{12}\,s_\phi^3}\big[\Pi\big(w \mid z\big) + \Pi\big(w; v \mid z\big)\big] + \frac{4\,c_{12}}{s_\phi c_\phi}\,,
\tag{29}
$$

where

$$
z = \frac{s_\phi^2}{s_{12}^2}\,, \quad v = \arcsin(c_{12}\sec\phi)\,, \quad w = \cot^2\theta_{12}\,\tan^2\phi\,,
$$

and K, E, Π and F are the standard elliptic functions [28]. The function $F2(\phi)$ reads

$$
F2(\phi) = -\frac{s_{12}^2}{s_\phi^2}\big[K(z_1) + F\big(v_1 \mid z_1\big)\big] + \frac{4}{s_\phi^2}\big[E(z_1) + E\big(v_1 \mid z_1\big)\big] +
$$

$$
\frac{4(c_\phi^2 - c_{12}^2)}{s_\phi^4}\big[\Pi\big(w_1 \mid z_1\big) + \Pi\big(w_1; v_1 \mid z_1\big)\big]\,,
\tag{30}
$$

where

$$
z_1 = \frac{1}{z}\,, \quad v_1 = \arcsin(c_\phi / c_{12})\,, \quad w_1 = \frac{1}{w}\,.
$$

Note, that in the regions, where the F1 (F2) function gives the contribution to Eq. (28), the following condition is always satisfied $z < 1$ ($z_1 < 1$). As concerns the quantity $I_\phi(\phi, c_{12} < 0)$, its analytical form coincides with (28) except the restrictions on the azimuthal angle , namely, in the upper row we have to write $0 < \phi < \pi - \theta_{12}$, in the middle row $\pi - \theta_{12} < \phi < \theta_{12}$, and in the bottom one $\theta_{12} < \phi < \pi$.

As we noted before, we are going to normalize the differential, with respect to the azimuthal angle ϕ, effects by the unpolarized corresponding quantities. Therefore, we need to calculate also a pure phase space integral $I(\phi, c_{12})$, and we write down it by the help of the functions

$$
F3(n\,, l\,, m) = \frac{2\,c_{12}\,c_\phi}{s_\phi^2}\big[F(l \mid m) - \Pi(n; l \mid m)\big]\,, \quad L = -\frac{1}{s_\phi^2}\ln\big(c_\phi^2 + c_{12}^2\,s_\phi^2\big)\,.
$$

If $c_{12} > 0$ we have

$$I(\phi, c_{12} > 0) = L + \begin{pmatrix} [F3(s_\phi^2, \theta_{12}, z) - 2\,F3(s_\phi^2, \pi/2, z)]/s_{12}, & 0 < \phi < \theta_{12} \\ [F3(s_{12}, \phi, 1/z) - 2\,F3(s_{12}, \pi/2, 1/z)]/s_\phi, & \theta_{12} < \phi < \pi - \theta_{12} \\ -F3(s_\phi^2, \theta_{12}, z)/s_{12}, & \pi - \theta_{12} < \phi < \pi + \theta_{12} \\ [F3(s_{12}, \phi, 1/z) - 2\,F3(s_{12}, \pi/2, 1/z)]/s_\phi, & \pi + \theta_{12} < \phi < 2\pi - \theta_{12} \\ [F3(s_\phi^2, \theta_{12}, z) - 2\,F3(s_\phi^2, \pi/2, z)]/s_{12}, & 2\pi - \theta_{12}\phi < 2\pi \end{pmatrix}.$$

(31)

For the case $c_{12} < 0$

$$I(\phi, c_{12} < 0) = L + \begin{pmatrix} -F3(s_\phi^2, \theta_{12}, z)/s_{12}, & 0 < \phi < \pi - \theta_{12} \\ -F3(s_{12}, \phi, 1/z)/s_\phi, & \pi - \theta_{12} < \phi < \theta_{12} \\ [F3(s_\phi^2, \theta_{12}, z) - 2\,F3(s_\phi^2, \pi/2, z)]/s_{12}, & \theta_{12} < \phi < 2\pi - \theta_{12} \\ F3(s_{12}, 2\pi - \phi, 1/z)/s_\phi, & 2\pi - \theta_{12} < \phi < \pi + \theta_{12} \\ -F3(s_\phi^2, \theta_{12}, z)/s_{12}, & \phi < \pi + \theta_{12} < \phi < 2\pi \end{pmatrix}. \quad (32)$$

In accordance with Eq. (22), the relation

$$\int_0^{2\pi} I(\phi, c_{12})\, d\phi = 4\pi$$

has to take place at any permissible values of c_{12}. We could not show this analytically but check this relation by means of the numerical integration. In this connection note that the quantities $I_{c_1}(\phi, c_{12})$, $I_{c_2}(\phi, c_{12})$ and $I_\phi(\phi, c_{12})$ satisfy also the conditions that can be deduced from a comparison of two different approaches to the angular integration given by Eqs. (6) and (18), namely

$$\int_0^{2\pi} \left[I_{c_1}(\phi, c_{12}) ; I_{c_2}(\phi, c_{12}) \right] d\phi = [\pi\, c_{12} ; \pi], \quad \int_0^{\pi} I_\phi(\phi, c_{12}) d\phi = \pi\, s_{12}.$$

4.3. Up-down differential asymmetries

In our paper [18] we found that in the rest system the angular distribution of the decay width, relative to the polar angle of the photon θ_2, provided the integration over the polar angle of the pion is performed, is trivial: it is proportional to c_2 if τ lepton is polarized and does not depend on this angle in unpolarized case.

There is just different situation if we are interesting in an azimuthal distribution. As we can see from the above results, even the pure phase space part, defined by Eq. (22), exhibits a nontrivial dependence on the angle θ_{12} (see also the angular region in Fig. 4). This dependence does not disappear after the integration over the angle θ_2, as it is seen from Eqs. (31) and (32). That is essential difference as compared with the polar angle distribution. Function $I(\phi, c_{12})$ is shown in Fig. 2 (right panel) for fixed positive and negative values of c_{12}.

To demonstrate this effect in details, we give in Figs. 5 - 9 the azimuthal distribution of the decay width, integrated over the variable c_2 in the upper hemisphere $(0 < \theta_2 < \pi/2;\ 0 < \phi < 2\pi)$, for both unpolarized and spin-dependent parts (the corresponding quantities are labeled by "up"). The spin-dependent part in these figures includes the contributions which are proportional to (Sq) and (Sk) and does not take into account the contribution proportional to (Spqk). The reason is that the last contribution, as well as the spin-independent part, is the same in the upper and lower $(\pi/2 < \theta_2 < \pi;\ 0 < \phi < 2\pi)$ hemispheres, whereas the first two terms are opposite in sign. It means that we can separate the contribution caused by (Sq) and (Sk) by taking the difference between the events in the upper and lower hemispheres (the corresponding quantities are labeled by "ud"). Because of the infrared divergence, in further we restrict ourselves by the condition $\omega > 0.3$ GeV, where the IB- and resonance contributions are of the same order. At small photon energies the IB-contribution dominates, and it is impossible to use the events in this region for the determination of the form factors.

In Fig. 5 we show the azimuthal distribution of the decay width, corresponding to the spin-independent part only, derived by a numerical integration over the pion and photon energies. We pay attention to a very strong sensitivity of this distribution to the parameter sets, that used to describe the structural resonance amplitude, in the wide range around $\phi = \pi$, where the IB-contribution has a minimum. We can conclude that the measurements in this region can be very important to discriminate between different theoretical models as well as between the parameter values used in these models.

The effects caused by the τ lepton the polarization due to contribution of the terms containing (Sq) and (Sk) are shown in Fig. 6. Together with the decay width we show here
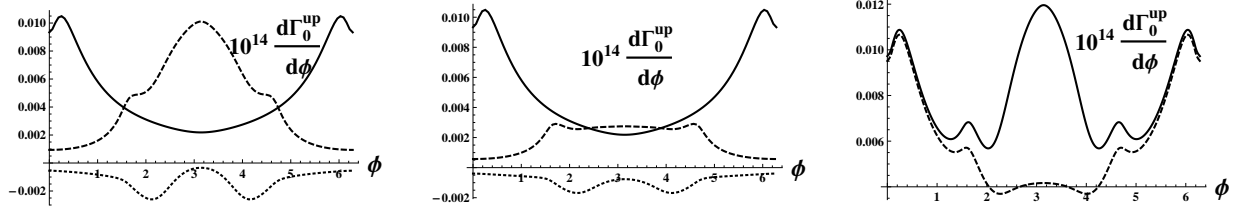
Figure 5. The spin-independent part of the differential decay width (in GeV· rad^{-1}), integrated over the variable c_2 in the upper hemisphere, versus the azimuthal angle. The left panel shows the IB-contribition (the solid line), the resonance contribution (the dashed line) and the IB-resonance interference (the dotted line) for the set 1 of the resonance parameters given in the Table 1; the middle panel is the same but for the set 2; the right panel shows the sum of all the contributions for the set 1 (the solid line), and the set 2 (the dashed line).

the polarization asymmetry defined as

$$A^{ud}(\phi) = \frac{d\,\Gamma_0^{up} + d\,\Gamma_0^{(s)up} - d\,\Gamma_0^{dn} - d\,\Gamma_0^{(s)dn}}{d\,\Gamma_0^{up} + d\,\Gamma_0^{(s)up} + d\,\Gamma_0^{dn} + d\,\Gamma_0^{(s)dn}} = \frac{d\,\Gamma_0^{(s)up}}{d\,\Gamma_0^{up}}, \tag{33}$$

where we labeled by "dn" the events in the lower hemisphere and used the symmetry relations

$$d\,\Gamma_0^{up} = d\,\Gamma_0^{dn}, \quad d\,\Gamma_0^{(s)up} = -d\,\Gamma_0^{(s)dn}.$$

Again, we see a strong sensitivity of both the spin-dependent decay width and the polarization asymmetry to the resonance parameter sets in the wide region around $\phi = \pi$.

In Fig. 7 (8) we present the azimuthal distributions for those spin-independent (spin-dependent) contributions to the partial decay width $d\,\Gamma_i$ which define the photon Stokes parameter ξ_i^{up} (the correlation parameters describing the influence of the τ lepton polarization on the photon Stokes parameters ξ_i^{ud}), $i = 1, 2, 3$. These partial decay widths are not defined positively. Note that the pure IB -contribution disappears for i=1.

Remind also that the parameters ξ_1 and ξ_3, which describe the linear polarization of the photon, depend on the choice of the photon polarization 4-vectors, and the parameter ξ_2, describing the circular polarization, does not depend.

In Fig. 9 we show the double distributions with respect to the angle ϕ and the invariant variable t for the up-down asymmetry and the correlation parameters. The corresponding integrated quantities $A^{ud}(\phi)$ and $\xi_i^{ud}(\phi)$ are given in Figs. 6 and 8, respectively.
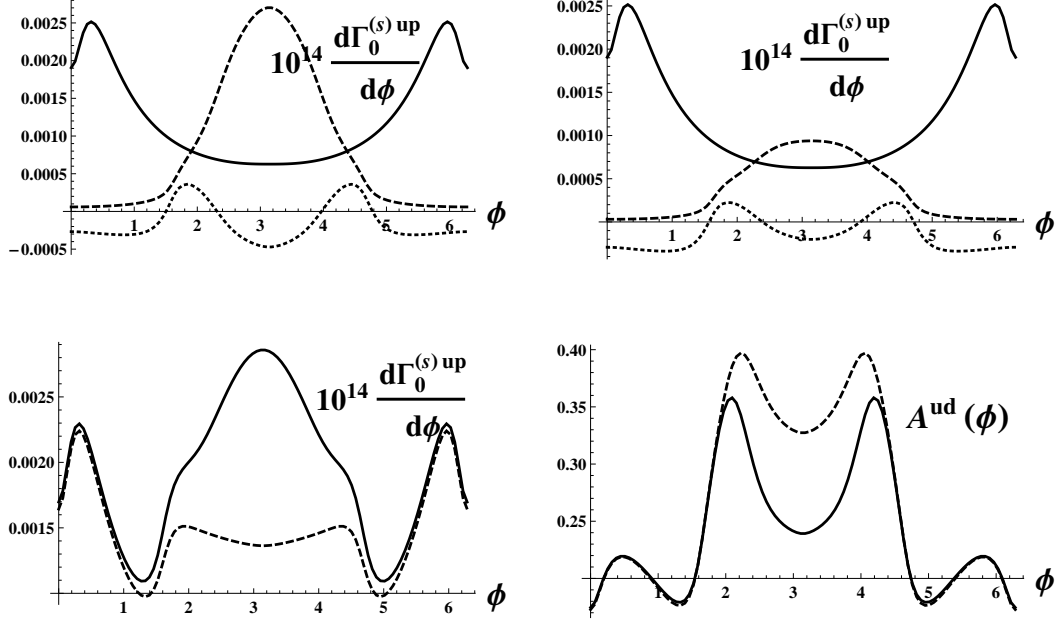
Figure 6. The quantities caused by the τ lepton polarization in the case of unpolarized photon. Notation for the quantities $d\Gamma_0^{(s)up}/d\phi$ are the same as in Fig. 5; the polarization asymmetry A^{ud} is calculated in accordance with Eq. (33) for the set 1 (the solid line) and the set 2 (the dashed line)of the parameters.

4.4. Right-left differential asymmetries

As we mention above, the azimuthal distribution, caused by the (Spqk) term in the differential decay width, can be separated by taking the difference between the events number in the right (R) $(0 < \theta_2 < \pi;\ 0 < \phi < \pi)$ hemisphere at fixed value of ϕ and in the left (L) $(0 < \theta_2 < \pi;\ \pi < \phi < 2\pi)$ one at the angle $2\pi - \phi$. The corresponding differences we labeled by "RL". So, we can define the corresponding asymmetry and the correlation parameters as

$$A^{RL}(\phi) = \frac{d\Gamma^R(\phi) - d\Gamma^L(2\pi - \phi)}{d\Gamma_0(\phi) + d\Gamma_0(2\pi - \phi)} = \frac{d\Gamma^R(\phi)}{d\Gamma_0(\phi)}, \quad \xi_i^{RL}(\phi) = \frac{d\Gamma_i^R(\phi)}{d\Gamma_0(\phi)}, \tag{34}$$

where $d\Gamma_0^{R(L)}(\phi)$ and $d\Gamma_i^R(\phi)$ are determined by the spin-dependent part (the term (Spqk)) of the $|M_\gamma|^2$, and $d\Gamma_0(\phi)$ – by the spin-independent one.

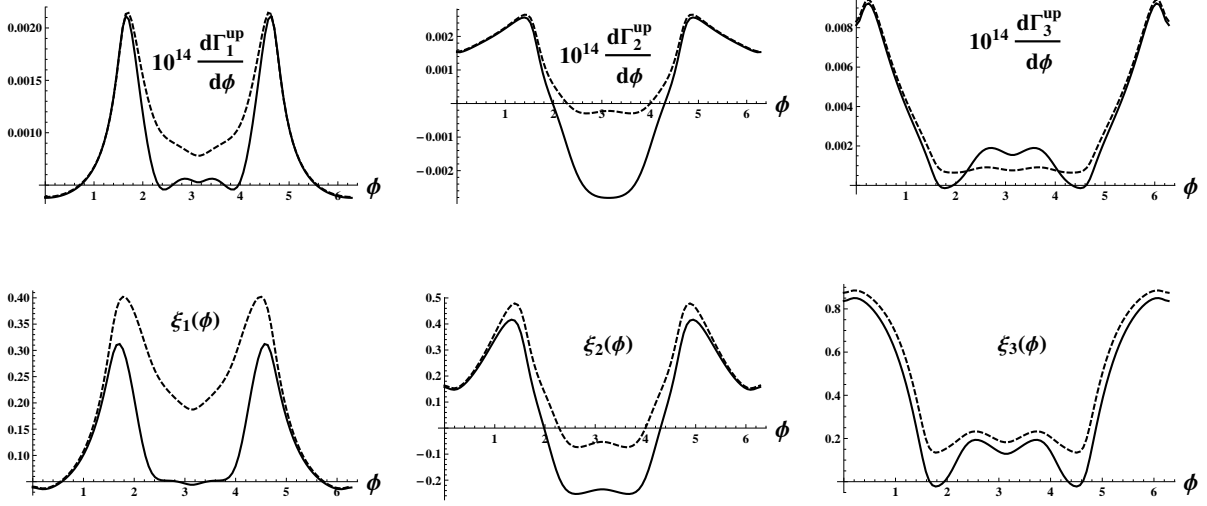In Figs. 10-12 we show some differential right-left asymmetries.

Figure 7. The partial decay widths (the upper row, in CeV· rad^{-1}) and the corresponding Stokes parameters (the lower row) are calculated for unpolarized τ lepton, in accordance with Eq. (5), in the upper hemisphere. The solid line corresponds to the set 1, the dashed line – to the set 2 of the parameters.
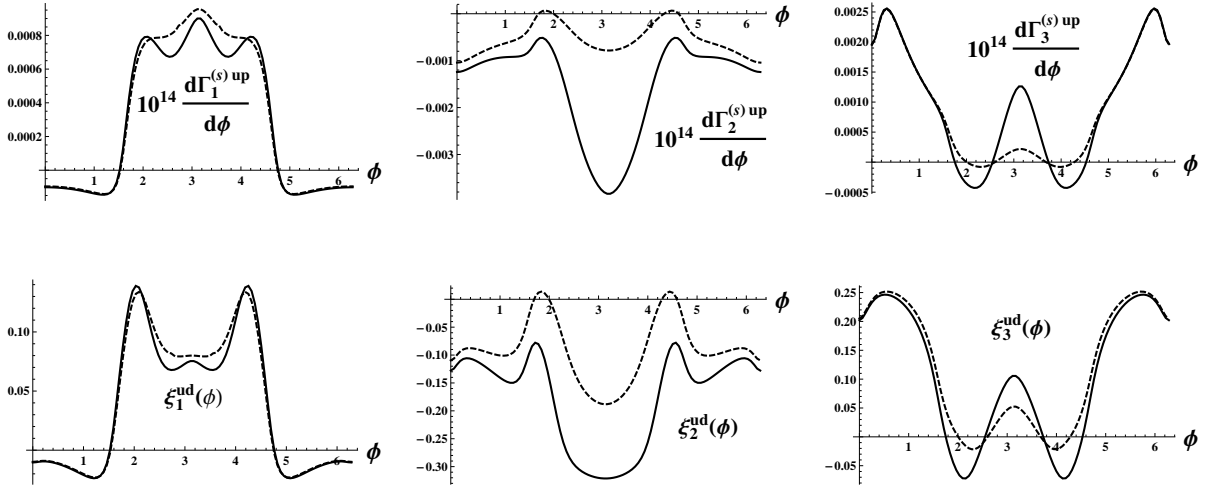

Figure 8. The same as in Fig. 7 but for the polarized τ lepton and for the difference of the corresponding events in the upper and lower hemispheres.

5. DISCUSSION

In this paper we investigated the photon angular distributions in the radiative decay of the polarized τ lepton. Special attention is paid to the study of the distribution over the photon azimuthal angle/ If τ is unpolarized, the squared matrix element depends on a pair of the dynamical variables only (the pion and photon energies, for example), and the angular part of the photon phase space in the coordinate system with the movable Z axis along the
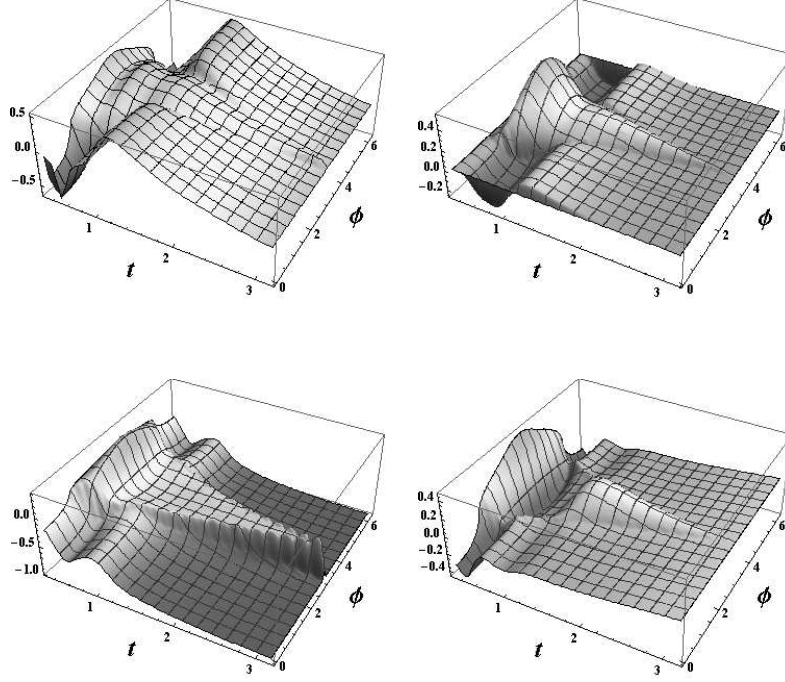
Figure 9. The double differential distributions for the up-down asymmetry $A^{ud}(t,\phi)$ (the left panel in the upper row) and the correlation parameters $\xi_1^{ud}(t,\phi)$ (the right panel in the upper row), $\xi_2^{ud}(t,\phi)$ and $\xi_3^{ud}(t,\phi)$ (the lower row) calculated with the set 2 of the parameters. The t-variable is given in GeV^2.
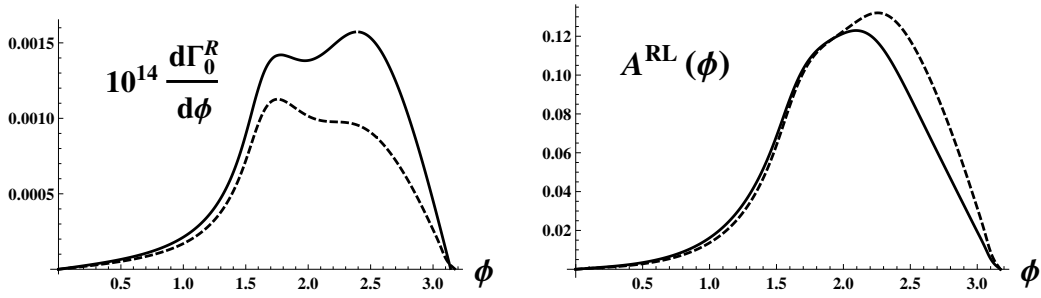

Figure 10. The decay width (in GeV·rad^{-1}) due to the terms proportional to (Spqk) in the right hemisphere and the right-left asymmetry defined by Eq. (33). The solid and dashed lines correspond to the set 1 and the set 2 of the parameters, respectively.

photon 3-momentum is fully factorized. In this case, the angular dependence of the decay width is absent. But in the system with fixed Z axis (along arbitrary direction) the photon angular phase space depends on the dynamical variables too, via the quantity

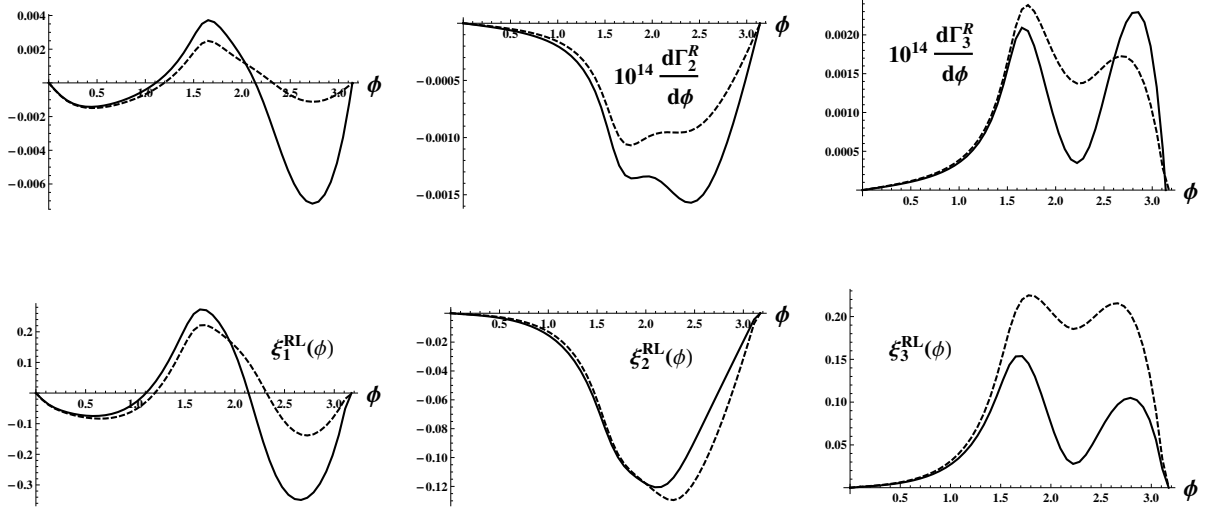$$c_{12} = \frac{M^2 + m^2 + 2\,\omega\,\epsilon - 2\,M(\omega + \epsilon)}{2\,\omega|\mathbf{q}|}$$

Figure 11. The partial decay width (in GeV·rad^{-1}) due to the terms proportional to (Spqk) in the right hemisphere and the right-left asymmetry defined by Eq.(33). The solid and dashed lines corresponds to the set 1 and the set 2 of the parameters, respectively.

(see Eqs. (6) and (18)). If we use the angular δ-function to perform the azimuthal integration, then only the double angular distribution is not trivial, because the integration with respect to any polar angle leads to the factorization of the residual part. This approach gives the possibility to study also some effects arising due to the τ lepton polarization (the terms containing (Sq) and (Sk) in $|M_\gamma|^2$). The corresponding double and single angular distributions can be calculated using Eqs. (7) and (9), respectively. Choosing the Z axis along the direction of the polarization vector, in the τ rest frame (see Fig. 2), we used this formalism in Ref. [18] to investigate the integral up-down effects with polarized τ lepton.

Using the similar approach, we can carry out the azimuthal integration in the right and left hemispheres separately, and study the difference of the corresponding quantities that caused by the spin-dependent terms proportional to (Spqk). We obtain the analytical expressions for the t-distribution of the integral (relative to the azimuthal angle) right-left asymmetries. In Fig. 3, we show the corresponding differential decay width (Eq. (12)) as well as the polarization asymmetry and the polarization parameters defined by Eq. (17). From Fig. 3 one can see that the effects considered have appreciable sensitivity to the parameters used for the description of the resonance amplitude, namely, to the vector and axial-vector form factors. For the differential decay width and the polarization parameters $\xi_1^{RL}(t)$ and $\xi_3^{RL}(t)$ such sensitivity manifests itself in the region t$\geq$0.6 GeV^2, whereas the polarization asymmetry $A^{RL}(t)$
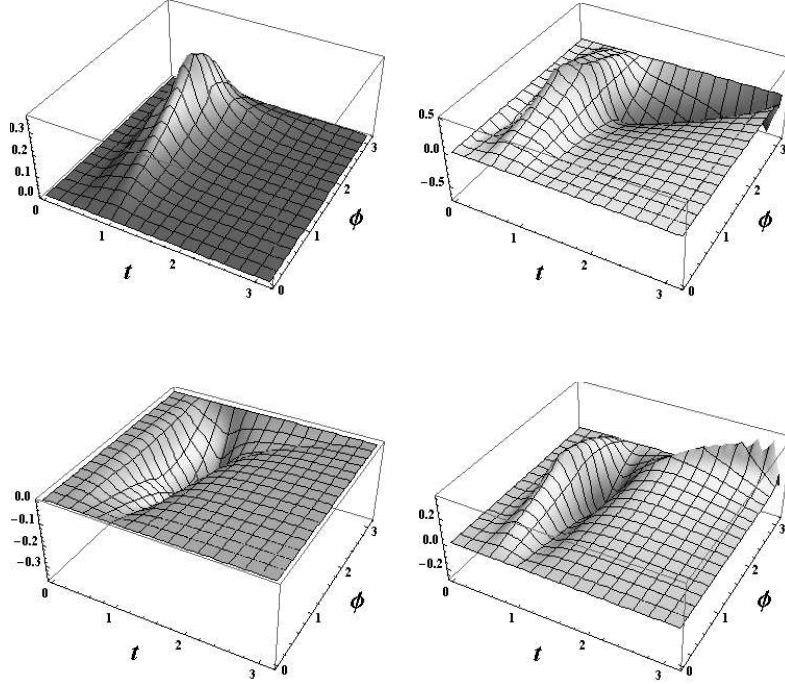
Figure 12. The same as in Fig. 9 but for the right-left asymmetry and the corresponding correlation parameters, calculated with the set 1 of the parameters+.

and the parameter $\xi_2^{RL}(t)$ are considerably different for the sets 1 and 2 of the parameters at $t \geq 1\,\mathrm{GeV}^2$. At such values of t, the resonance amplitude M_R can dominate.It means that the integral (with respect to the azimuthal angle) right-left asymmetries can be used to study the model-dependent parameters used for the description M_R, particularly the vector and axial-vector form factors.

We can also keep the azimuthal dependence of the observables and use the δ-function to perform the integration over the pion polar angle. In this case, the residual phase space factor is more complicated. The variation limits of the photon polar (θ_2) and azimuthal (ϕ) angles are defined by Eqs. (20), (21) and are shown in Fig. 4. They depend essentially on the absolute value and sign of the quantity c_{12} and on the solution for c_1 in the relation (18). The further integration over c_2 is performed analytically for both spin-dependent and spin-independent contributions in $|M_\gamma|^2$. Somewhat unexpected result is that even the azimuthal dependence of the unpolarized contribution has a nontrivial structure which connected directly with the quantity $I(\phi, c_{12})$ defined by Eqs. (31), (32) and it is shown in Fig. 2 for the positive and negative values of c_{12} (the right panel). The positions of the sharp maxima of the function $I(\phi, c_{12})$, which depend on c_{12}, point to the enhancement of the

events number at the corresponding values of the angle ϕ. Because the IB - and resonance amplitudes in M_γ have very different dependence on the pion and photon energies (and on c_{12} too), we think that the azimuthal distribution of the decay width and of the different polarization observables can be useful to probe the model-dependent resonance contribution. This statement is confirmed by the illustration of the differential up-down (Figs. 5-9) and right-left (Figs. 10-12) asymmetries in the decay (1). The curves in these figures are obtained by the integration with respect to the pion and photon energies taking into account the events with $\omega > 0.3$ GeV. This restriction eliminates the events with small photon energies, where the IB-mechanism dominates due to the infrared divergence, and it allows to study more reliably the resonance mechanism.

In Fig. 5 (6) we present the spin-independent (the spin-dependent) parts of the decay width and the corresponding polarization asymmetry for the events in the upper hemisphere ($c_2 > 0$). Firstly, let us pay attention to the high sensitivity of these observables to the model parameters that manifest itself by the strong distinction between the curves in Fig. 5 (the right panel) and in Fig. 6 (the lower row), which correspond to the set 1 and the set 2 of the parameters. Besides, we note the suppression of the IB-contribution and the enhancement of the resonance one for the set 1 of the parameters in the wide region around $\phi = \pi$. These remarks remain valid also for the Stokes (Fig. 7) and the correlation (Fig. 8) parameters, though we do not give separately the contributions of the corresponding amplitudes and their interference (as in Figs. 5,6). The Stokes parameters ξ_1 and ξ_2 as well as the correlation parameter ξ_2^{ud} show a high model dependence.

In Fig. 9 we demonstrate the double distribution over the t and ϕ variables for the polarization asymmetry and the correlation parameters. The integration over the azimuthal angle in the numerators and denominators of the expressions, which define these quantities (see Eq. (5)), allows to calculate the t-dependencies of these observables obtained in Ref. [18] in an analytical form. We check statement by the numerical integration over the ϕ variable. Remind that the up-down effects are determined by the difference of the events with the photon in the upper and lower hemispheres, and they are symmetrical under the change $\phi \to 2\pi - \phi$. These effects arise due to the terms proportional to (Sq) and (Sk) in $|M_\gamma|^2$.

The right-left effects, caused by the difference of the events in the right and left hemispheres, are antisymmetrical under this change and arise due to the terms proportional to (Spqk). Some of them are presented in Figs. 10-12. We can see that they are several time

smaller in absolute value as compared with the up-down effects. The quantities ξ_1^{RL} and ξ_3^{RL}, which describe linear polarization of the photons, show a strong dependence on the model-dependent parameters whereas the parameter of the circular polarization ξ_2^{RL} and the polarization asymmetry A^{RL} do not show such dependence. Again, by the integration over the ϕ variable of the double distributions (over the t and ϕ variables), we have to calculate the curves given in Fig. 3 which correspond to our analytical results for the integral left-right effects (Eqs. (7-10)). We checked this statement by the numerical integration.

In this paper, we mainly analyse the observables with the large photon energies ($\omega >$ 0.3 GeV) when the values of the IB- and resonance amplitudes are of the same order. The measurements in this region allows to study the model-dependent vector and axial-vector form factors. In the region of the small photon energies (up to 0.1 GeV) the IB-contribution dominates, and the uncertainty of different differential decay widths caused by the form factors is of a few percent. Thus, the measurements of the A^{ud} or A^{RL} asymmetries in this region can be used, in principle, to determine the τ-lepton polarization degree.

6.   CONCLUSION

The radiative one-meson decay of the polarized τ lepton, $\tau^- \to \pi^- \gamma \nu_\tau$, has been investigated. The presence of the arbitrarily oriented 3-vector of the τ lepton polarization leads to the azimuthal dependence of the emitted photon which is absent if the τ lepton is unpolarized. So, we pay special attention to the investigation of the various distributions over the photon azimuthal angle. In connection with this, the photon phase space is discussed in more detail since in the case of the polarized τ lepton it is nontrivial and, therefore, it requires of thorough investigation and, as we know such analysis is absent in the literature. We think that this detailed investigation of the angular part of the three-body phace space can be useful in the analysis of various angular distributions in the three-body decay of the polarized particles. The azimuthal dependence of the following polarization observables has been calculated: the asymmetry caused by the τ lepton polarization, the Stokes parameters of the emitted photon and the spin correlation coefficients which describe the influence of the τ lepton polarization on the photon Stokes parameters.

The amplitude of the τ lepton decay, $\tau^- \to \pi^- \gamma \nu_\tau$, has two contributions: the inner bremsstrahlung, which does not contain any free parameters, and the structure-dependent

term which is parameterized in terms of the vector and axial-vector form factors. Note that in our case these form factors are the functions of the t variable and $t > 0$, i.e., we are in the time-like region. The form factors, in this region, are the complex functions and their full determination, that is to say, not only of their moduli but their phases as well, is non-trivial in this case. To do this it is necessary to perform the polarization measurements.

The calculation of various observables was done for two sets of the parameters describing the vector and axial-vector form factors. The numerical estimation shows that some polarization observables can be effectively used for the discrimination between two parameter sets since these observables significantly differs in some regions of the photon azimuthal angle.

We found that the investigation of the azimuthal distributions of the different observables in the radiative decay of the polarized τ lepton including the decay width, the polarization asymmetry, the Stokes and the correlation parameters of the photon itself is very fruitful for the analysis of the phenomenological models describing the hadronization of the weak charged currents.

7. ACKNOWLEDGMENTS

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